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MODULES AND GOLOD HOMOMORPHISMS

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Introduction

If (R, m, k) is a local ring and M a finitely generated R-module, we denote by P_R^M the Poincaré series of M over R. This is the series $\sum_{i=0}^{\infty} B_i z^i$ where $B_i = \dim_k \operatorname{Tor}_i^R(M, k)$. For any homomorphism $(R, m, k) \to (S, n, k)$ of local rings, there is a change of rings spectral sequence:

$$\operatorname{Tor}_p^S(M, \operatorname{Tor}_q^R(S, k)) \Rightarrow \operatorname{Tor}_{p+q}^R(M, k)$$

From this one obtains a coefficient-wise inequality of power series

$$P_S^M \le P_R^M G_S^R \quad \text{where } G_S^R = (1 - z(P_R^S - 1))^{-1}$$

We will call M an f-Golod module if equality holds and $n\Gamma\operatorname{Tor}^R(S, k) = 0$. If k is f-Golod, then f is called a Golod homomorphism. (In this case the condition $n\Gamma\operatorname{Tor}^R(S, k) = 0$ is redundant [4, Theorem 4.6]). Golod homomorphisms have been studied in [8], [9], and [4] and f-Golod modules were considered in [9].

A recent theorem of Avramov [4, Theorem 2.3] makes it possible to say some new things about f-Golod modules. In particular, an f-Golod module can only occur if f is Golod. This is demonstrated in Section 1 along with some characterizations and examples of f-Golod modules. In keeping with the recent homotopical spirit of this subject, Section 2 further characterizes f-Golod modules using only Golod homomorphisms of rings and differential graded algebras.

1. Characterizations by syzygies

We will assume throughout this paper that modules are finitely generated. For references on Massey products see [1] or [9]. If M is an S-module, let

$$\cdots \to X_i \xrightarrow{d_i} \cdots \to X_0 \xrightarrow{d_0} M \to 0$$

be a minimal resolution of M by free S-modules. The i-th syzygy of M is $L_i = \operatorname{Im} d_i$.

Now suppose $f: R \to S$ is a homomorphism of local rings and M is an S-module different from 0.

1.1. Theorem. *The following are equivalent:*

(i) *The S-module M is f-Golod.*

(ii) *The homomorphism f is Golod and the induced map f_*: $\operatorname{Tor}^R(M,k) \to \operatorname{Tor}^S(M,k)$ is injective.*

(iii) *The induced maps*

$$\operatorname{Tor}^R(M,k) \to \operatorname{Tor}^S(M,k) \quad \textit{and} \quad \operatorname{Tor}^R(L_1,k) \to \operatorname{Tor}^S(L_1,k)$$

are injective, where L_1 is the first syzygy of M as an S-module.

(iv) *For every $i>0$, the i-th syzygy L_i of M as an S-module is f-Golod.*

Proof. When f is one of Avramov's exceptional homomorphisms [4], an easy calculation shows that conditions (i) through (iv) are vacuous. Otherwise, we will prove the equivalences in the following sequence: (i) $\Rightarrow$ (iv) $\Rightarrow$ (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i).

Proof of (i) $\Rightarrow$ (iv). It suffices to show that if an S-module N is f-Golod, then so is its first syzygy L. Let Y be a minimal resolution of M by free S-modules and X a minimal resolution of k by fee R-modules. Put $(tY)_i = Y_{i+1}$ for $i>0$. Then tY is a minimal resolution of L. The spectral sequence

$$\text{(A)} \qquad E^2_{p,q} \simeq \operatorname{Tor}^S_p(N, \operatorname{Tor}^R_q(S,k)) \Rightarrow \operatorname{Tor}^R_{p+q}(N,k)$$

comes from filtering the double complex $Y \otimes_R X$ while the spectral sequence

$$\text{(B)} \qquad E'^2_{p,q} \simeq \operatorname{Tor}^S_p(L, \operatorname{Tor}^R_q(S,k)) \Rightarrow \operatorname{Tor}^R_{p+q}(L,k)$$

comes from filtering the double complex $(tY) \otimes_R X$. Since N is f-Golod, all the differentials $d^r_{p,q} = 0$ for $q>0$ and $r>2$, and $E^\infty_{p,q} = 0$ for $q>0$ in spectral sequence(A). Starting with the isomorphisms $E^0_{p,q} \simeq E'^0_{p-1,q}$ for $p>0$, one may argue that the same properties hold for spectral sequence (B). It follows that L is f-Golod. This argument follows a suggestion of J.-E. Roos and C. Löfwall and may be found in more detail in [9, Theorem 1.3].

Proof of (iv) $\Rightarrow$ (iii). In the spectral sequence (A) above, the edge homomorphism

$$\operatorname{Tor}^R_p(M,k) \xrightarrow{j} E^\infty_{p,0} \to E^2_{p,0} \xrightarrow{\sim} \operatorname{Tor}^S_p(M, S \otimes_R k) \xrightarrow{\phi} \operatorname{Tor}^S_p(M,k)$$

is the map f_* induced by f. Since M is f-Golod, ϕ is injective and $E^\infty_{p,q} = 0$ for $q>0$, so j is an isomorphism and then f_* is injective.

Proof of (iii) $\Rightarrow$ (ii). For a differential graded algebra A, augmented over k, and a differential graded A-module P, $M(H(A), H(P))$ denotes the set of all elements of $H(P)$ decomposable as matric Massey products $\{x_1, \dots, x_r, y\}$ where $x_i \in IH(A)$ and $y \in H(P)$. By [4, Theorem 2.3] the map $R \to S$ is a Golod homomorphism if $I\operatorname{Tor}^R(S,k)$ has trivial Massey operations. From the diagram

$$\begin{tikzcd}
\dots \arrow[r] & \mathrm{Tor}_p^R(L,k) \arrow[r] & \mathrm{Tor}_p^R(Y_0,k) \arrow[r] & \mathrm{Tor}_p^R(M,k) \arrow[r] \arrow[d] & \mathrm{Tor}_{p-1}^R(L,k) \arrow[r] \arrow[d] & \dots \\
 & & & \mathrm{Tor}_p^S(M,k) \arrow[r, "\sim"] & \mathrm{Tor}_{p-1}^S(L,k) \arrow[r] & \dots
\end{tikzcd}$$

for $p>0$, one sees that $\mathrm{Tor}_p^R(L,k)\to\mathrm{Tor}_p^R(Y_0,k)$ is surjective. Thus every Massey product in $M(\mathrm{Tor}^R(S,k),\mathrm{Tor}_+^R(Y_0,k))$ is the image of a Massey product in $M(\mathrm{Tor}^R(S,k),\mathrm{Tor}^R(L,k))$. But by [8, Theorem 2.10], $M(\mathrm{Tor}^R(S,k),\mathrm{Tor}^R(L,k))$ is precisely the kernel of the induced mapping $\mathrm{Tor}^R(L,k)\to\mathrm{Tor}^S(L,k)$. Since Y_0 is a free S-module of positive rank, $M(\mathrm{Tor}^R(S,k), I\mathrm{Tor}^R(S,k))=0$ and f is Golod.

Proof of (ii) ⇒ (i). Here it is convenient to use the Eilenberg–Moore spectral sequence as formulated by Avramov in [2, Theorem 3.1.1]:

$$\mathrm{Tor}_{p,q}^{\mathrm{Tor}^R(S,k)}(\mathrm{Tor}^R(M,k),k)\Rightarrow \mathrm{Tor}_{p+q}^S(M,k)$$

Since f is Golod every Massey product $\{x_1,\dots,x_r\}=0$, where $x_i\in I\mathrm{Tor}^R(S,k)$, $i=1,\dots,r$. Now, since f_* is injective, every Massey product $\{x_1,\dots,x_r,y\}=0$ where $y\in\mathrm{Tor}^R(M,k)$ and $x_i\in I\mathrm{Tor}^R(S,k)$, $i=1,\dots,r$. It is known [5], [1] that the differentials in the above spectral sequence are given by Massey products. So the sequence degenerates yielding the desired identity of Poincaré series.

1.2. Examples of f-Golod modules

(1) If $f: R\to S$ is a Golod homomorphism, then the residue field k and all of its syzygies as an S-module are f-Golod (Theorem 1.1 (iv)).

(2) If R is a local ring, M a finitely generated R-module, and x a non zero-divisor in $m(\mathrm{ann}\, M)$, then M is $R\to R/(x)$-Golod [10].

(3) Let $f:(R,m,k)\to(S,n,k)$ be a homomorphism of local rings and X a free resolution of k. Then f is strong Golod if there is a mapping of complexes $H(X\otimes_R S)\to X\otimes_R S$ inducing an isomorphism in homology. Here $H(X\otimes_R S)$ is regarded as a complex with trivial differential. From [9, Theorem 4.6] we have: If $f: R\to S$ is strong Golod and M is a finitely generated S-module such that $(0:n)M=0$, then M is f-Golod. In particular, every proper ideal of S is f-Golod. There are lots of strong Golod homomorphisms. In fact for any local ring R, $R\to R/m^i$ is strong Golod for large i [8, Theorem 3.17].

(4) If (R,m,k) is a local ring and M a finitely generated R-module, then the trivial extension of R by M is the ring $R(M)=R+M$ with multiplication $(r,a)(s,b)=(rs, rb+sa)$. A theorem of Gulliksen [6] shows that $R\to R(M)$ is a Golod homomorphism so that

$$P_{R(M)}^k = P_R^k/(1-zP_R^M).$$

On the other hand, Herzog [7] has shown that for any R-module N, regarded as an $R(M)$-module via the map $R(M) \to R$,

$$P^N_{R(M)} = P^N_R P^R_{R(M)}.$$

In particular $P^k_{R(M)} = P^k_R P^R_{R(M)}$. Combining this with Gulliksen's formula shows that N is f-Golod.

2. Characterizations by homomorphisms of rings

In the last few years the homological theory of local rings has been enriched by a new topological language, mirroring parallel structures in rational homotopy theory. For example, if $f: R \to S$ is a homomorphism of local rings, and X is an algebra resolution of k, then $X \otimes_R S$ is called the 'fibre' of f. It is unique up to a 'homotopy equivalence' and there is a homotopy exact sequence connecting the homotopy Lie algebras of R, S, and the fibre. It has not been clear, however, what role modules are to play in this scenario, so we recast the idea of an f-Golod module using only homomorphisms of rings.

2.1. Theorem. *Let $f: (R, m, k) \to (S, n, k)$ be a homomorphism of local rings and M a finitely generated S-module. Let F_R be the fibre of the map $R \to R(M)$ and F_S the fibre of the map $S \to S(M)$. Then the following are equivalent.*

(i) *The S-module M is f-Golod*

(ii) *The composite $R \to S \to S(M)$ is a Golod homomorphism.*

(iii) *f is Golod and the induced map of fibres $F_R \to F_S$ is Golod*

If f is surjective, then the following condition is also equivalent:

(iv) *The map $R(M) \to S(M)$ is Golod.*

Proof. *Proof of* (i) $\Leftrightarrow$ (iii). Let X be an R-algebra resolution of k. Then $F_R = X \otimes_R R(M)$. By [4, Theorem 2.3], since $R \to R(M)$ is Golod, there is a homotopy equivalence

$$F_R \sim k(IH(F_R)) \simeq k(\mathrm{Tor}^R(M, k)).$$

Similarly, if Y is a minimal S-algebra resolution of k, then $F_S \sim k(\mathrm{Tor}^S(M, k))$. Thus $P^k_{F_R} = (1 - zP^M_R)^{-1}$, $P^k_{F_S} = (1 - zP^M_S)^{-1}$, and

$$\mathrm{Tor}^{F_R}(F_S, k) \approx \mathrm{Tor}^{k(\mathrm{Tor}^R(M,k))}(k(\mathrm{Tor}^S(M, k)), k).$$

The latter may be calculated using the bar construction

$$\begin{aligned} U &= \bar{B}(k(\mathrm{Tor}^R(M, k)), k(\mathrm{Tor}^S(M, k))) \\ &\simeq T(s\mathrm{Tor}^R(M, k)) \otimes k(\mathrm{Tor}^S(M, k)) \end{aligned}$$

where T denotes the tensor algebra.

For $(t, y) \in k(\mathrm{Tor}^S(M, k))$, $x_i \in \mathrm{Tor}^R(M, k)$, $i = 1, \ldots, r$,

$$d([x_1|\dots|x_r|(t,y)]) = [x_1|x_2|\dots|x_{r-1}]tx_r.$$

Hence $H(U) \approx T(s\mathrm{Tor}^R(M,k)) \otimes k(\mathrm{Coker}\, f_*)$.

If M is f-Golod, f_* is injective, so

$$P_{F_R}^{F_S} - 1 = (P_S^M - P_R^M)/(1 - zP_R^M).$$

Then one checks easily that $F_R \to F_S$ is Golod. That f is Golod was shown in Theorem 1.1.

To reverse the argument, note that the Poincaré series identity implies that the Hilbert series of Coker f_* is $P_S^M - P_R^M$ indicating that f_* is injective.

Proof of (i) ⇔ (ii). First assume that M is f-Golod. By Theorem 1.1, f is Golod. Consequently,

$$P_S^M = P_R^M/(1 - zP_R^S) \quad \text{and} \quad P_S^k = P_R^k G_S^R.$$

So $P_{S(M)}^k = P_S^k/(1 - zP_S^M) = P_R^k G_{S(M)}^R$ and $R \to S(M)$ is Golod.

Now assume that $R \to S(M)$ is Golod. Then $\Pi\mathrm{Tor}^R(S(M),k)$ has trivial Massey products, but since the induced map $\mathrm{Tor}^R(S,k) \to \mathrm{Tor}^R(S(M),k)$ is injective, $\Pi\mathrm{Tor}^R(S,k)$ also has trivial Massey products, so f is Golod. A calculation with Poincaré series shows that M is f-Golod.

Proof of (iii) ⇔ (iv) *if f is surjective.* Let $J = \mathrm{Ker}\, f = \mathrm{Ker}(R(M) \to S(M))$. Then J is an ideal in $R(M)$ on which M acts trivially, so by (1.2) J is $R \to R(M)$-Golod. Since f is surjective

$$P_{R(M)}^{S(M)} - 1 = zP_{R(M)}^J = zP_R^J/(1 - zP_R^M) = (P_R^S - 1)/(1 - zP_R^M).$$

It follows that $P_{R(M)}^k G_{S(M)}^{R(M)} = P_R^k G_{S(M)}^R$, so $R \to S(M)$ is Golod if and only if $R(M) \to S(M)$ is Golod.

The equivalence of (i) and (iv) is a sharper form of an unpublished theorem of the author and J.-E. Roos. Namely, we proved that if f is surjective, then $R(M) \to S(M)$ is Golod if and only if $f: R \to S$ is Golod and M is f-Golod. We also proved the following complementary result:

Suppose that R is a local ring and M and N are finitely generated R-modules. Then if $g: M \to N$ is a homomorphism of R-modules, the induced map $g_*: \mathrm{Tor}^R(M,k) \to \mathrm{Tor}^R(N,k)$ is injective if and only if $R(M) \to R(N)$ is a Golod homomorphism.

This strengthens Avramov's result [1] that g_* is injective if and only if $R(M) \to S(M)$ is small.

References

[1] L.L. Avramov, Small homomorphisms of local rings, J. Algebra 50 (1978) 400–453.

[2] L.L. Avramov, Obstructions to the existence of multiplicative structures on minimal free resolutions, Amer. J. Math. 103 (1981) 1–31.

[3] L.L. Avramov, Local algebra and rational homotopy, in: J.-M. Lemaire and J.-C. Thomas, eds., Homotopie Algébrique et Algèbre Locale, Astérisque 113-114 (1984) 15-43.

[4] L.L. Avramov, Golod homomorphisms, in: J.-E. Roos, ed., Algebra, Algebraic Topology, and their Interactions. Lecture Notes in Math. (Springer, Berlin, to appear).

[5] V.K.A.M. Gugenheim and J.P. May, On the theory and applications of differential torsion products. Memoirs Amer. Math. Soc. 142 (1974).

[6] T.H. Gulliksen, Massey operations and the Poincaré series of certain local rings, J. Algebra 22 (1972) 223-232.

[7] J. herzog, Algebra retracts and Poincaré series, Manuscripta Math. 21 (1977) 307-314.

[8] G. Levin, Local rings and Golod homomorphisms, J. Algebra 37 (1975) 266-289.

[9] G. Levin, lectures on Golod homomorphisms, Preprint Series, Dept. of Math., Univ. of Stockholm, No. 15, 1976.

[10] J. Shamash, The Poincaré series of a local ring, J. Algebra 12 (1969) 453-470.